Complexes, cones, and triangles

Let \mathcal{A} be an abelian category, and let $\mathcal{C}(\mathcal{A})$ denote the category of (cochain) complexes in \mathcal{A} and morphims of complexes. Thus an object of \mathcal{C} is a sequence of composable maps:

$$\cdots \to A^{q-1} \xrightarrow{d^{q-1}} A^q \xrightarrow{d^q} A^{q+1\cdots}$$

such that $d^q \circ d^{q-1} = 0$ for all $q \in \mathbf{Z}$. By definition, if (A, d) is a complex, the complex A[m] is defined by

$$A[m]^q := A^{q+m}$$
, and $d^q_{A[m]} := (-1)^m d^{q+m}$

Note that (A[m])[n] = A[m+n] (this is actually an equality, not an isomorphism!). Let $u: A^{\cdot} \to B^{\cdot}$ be a morphism in \mathcal{C} . Then the *cone* of u is the complex $(C^{\cdot}(u), d_C)$, where

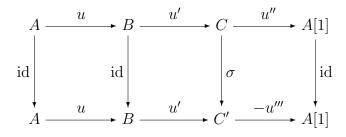
$$C^q := B^q \oplus A^{q+1}$$
 and $d_C(b, a) := (db - u(a), -da).$

Some authors use instead the complex with the same terms but with $d_{C'}(b, a) := (db + u(a), -da)$. In fact the map $\sigma: (C, d_C) \to (C', d_{C'})$ sending (b, a) to (b, -a) is an isomorphism of complexes.

There is an exact sequence of complexes:

$$0 \longrightarrow B \xrightarrow{u'} C \xrightarrow{u''} A[1] \longrightarrow 0,$$

where u'(b) := (b, 0) and u''(b, a) := a. Note that there is a commutative diagram:



if u''' in the bottom row is also defined by u'''(b, a) = a. Finally, observe that C(u[1]) = (C'(u))[1].

A *triangle* in \mathcal{C} is a sequence of three composable maps:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1].$$

A morphism of triangles is a commutative diagram:

$$\begin{array}{cccc} A & \underbrace{u} & B & \underbrace{v} & C & \underbrace{w} & A[1] \\ f & g & & & \\ f & g & & & \\ A' & \underbrace{u'} & B & \underbrace{v'} & C' & \underbrace{w'} & A[1] \end{array}$$

Note that the data of the triangle includes its starting point A. In particular, note that the triangle gives a sequence of maps

$$\cdots \longrightarrow B[-1] \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow A[1] \longrightarrow B[1] \longrightarrow \cdots,$$

but this sequence does not determine the triangle. Observe that if one changes the sign of two of the arrows in a triangle, the new triangle thus obtained is isomorphic to the original triangle, but this is not necessarily the case if one changes one or three of the signs. In particular, if C(u) is the mapping cone of a morphism $u: A \to B$, then C(u[1]) = C'(u)[1] and is isomorphic to C(u)[1], but the triangle starting at A[1] deduced from u[1] involves an odd number of sign changes when compared to the triangle obtained by shifting the triangle starting at A deduced from u.

Recall that two maps of complexes $f, g: A^{\cdot} \to B^{\cdot}$ are *homotopic* if there exists a family of maps $R^q: A^q \to B^{q-1}$ such that g - f = dR + Rd.

Definition 1 Let $\mathcal{K}(\mathcal{A})$ denote the homotopy category of complexes in \mathcal{A} . A triangle in \mathcal{K} is distinguished if it is isomorphic to a triangle of the form:

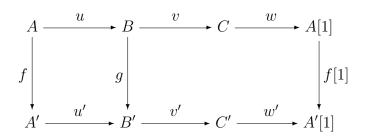
$$A \xrightarrow{u} B \xrightarrow{u'} C(u) \xrightarrow{u''} A[1]$$

for some morphism $u \in C(\mathcal{A})$. A triangle is antidistinguished if is obtained from a distinguished triangle by changing the sign of one of the morphisms.

Theorem 2 The set DT of distinguished triangles in \mathcal{K} satisfied the following conditions.

- 1. (TR1)
 - (a) Every triangle which is isomorphic to a distinguished triangle is distinguished.

- (b) Every morphism can be embedded in a triangle.
- (c) For any A, $A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$ is distinguished.
- 2. (TR2)
 - (a) The shift [1] of any distinguished triangle is antidistinguished.
 - (b) A triangle $A \to B \to C \to A[1]$ is distinguished if and only if $B \to C \to A[1] \to B[1]$ is antidistinguished.
- 3. (TR3) Given two distinguished triangles and a diagram:



there is a morphism $h: C \to C'$ (not necessarily unique!) so that (f, g, h) is a morphism of triangles.

4. (TR4) The octahedral axiom.

Proof: To prove part (c) of TR1, we have to check that the mapping cone C of the identity map of A is homotopic to 0. In other words, we have to prove that id_C is homotopic to 0. Recall that $C^q = A^q \oplus A^{q+1}$ with $d_C(a, a') = (da - a', -da')$. Then if R(a, a') := (0, -a),

$$(dR + Rd)(a, a') = d(0, -a) + R(da - a', -da') = (a, da) + (a' - da) = (a, a').$$

We have already discussed (a) of TR2. For (b), suppose $u: A \to B$ is a morphism in \mathcal{C} and let

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

be the associated distinguished triangle. Let C(v) denote the mapping cone

of v. We claim that there is a commutative diagram in \mathcal{K} :

where g and h are inverses of each other. Recall that

$$C(v)^q = C^q \oplus B^{q+1} = B^q \oplus A^{q+1} \oplus B^{q+1},$$

with d(b, a, b') = (db - u(a) - b', -da', -db') and define g(a) := (0, a, -u(a))and h(b, a, b') := a. We check that dh(b, a, b') = -da and hd(b, a, b') = h(db - u(a) - b', -da, -db') = -da so h is a morphism. Similarly,

$$dg(a) = d(0, a, -u(a)) = (-u(a) - u(a), -da, du(a)) = (0, -da, du(a))$$

and

$$g(d_{A[1]}a) = g(-d_Aa) = (0, -da, u(da))$$

Note that the diagram commutes. Finally, check that hg = id and gh is homotopic to the identity, with homotopy given by

$$R(b, a, b') := (0, 0, -b).$$

Conversely, suppose that

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is a triangle such that

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$$

is distinguished. Applying what we have just proved twice, we conclude that

$$A[1] \xrightarrow{-u[1]} B[1] \xrightarrow{-v[1]} C \xrightarrow{-w[1]} A[2]$$

is distinguished. Since we have changed the sign three times, it follows that

$$A[1] \xrightarrow{u[1]} B[1] \xrightarrow{v[1]} C \xrightarrow{w[1]} A[2]$$

is antidistinguished, and hence that the original triangle is distinguished.

To prove (3), we may assume that the two triangles come as the cones of morphisms u and u' respectively. By assumption, the square in the diagram is commutative in the homotopy category, so that there is a family of maps $R^q: A^q \to B'q - 1$ such that gu = u'f + Rd + dR. Now define:

$$h: C^q \to C'^q: (b,a) \mapsto (g(b) - R(a), f(a))$$

Then it is clear that f[1]w = w'h and that v'g = hv. We still need to check that h is a morphism of complexes.

$$dh(b,a) = d(g(b) - R(a), f(a)) = (dg(b) - dR(a) - u'f(a), -df(a))$$

$$hd(b,a) = h(db - u(a), -da)$$

= $(g(db) - gu(a) + Rda, -fda)$

Thus it suffices to observe that -dR(a) - u'f(a) = -gu(a) + Rda.